Shock waves in a dusty gas

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SUMMARY

The plane steady decelerated flow of a dust–gas mixture is analysed in an approximate manner. The problem, which has a five-parameter family of solutions, is reduced to a form such that the analysis can be completed by the integration of a first-order non-linear differential equation and a quadrature. A few integral curves are given and the characterizing features of the flow field are discussed.

1. INTRODUCTION

When a shock wave is propagated through a gas which contains an appreciable amount of dust, the thickness of the wave, the pressure change across the shock, and the other features of the flow differ greatly from those which arise when the shock passes through a dust-free gas. We consider here the stationary plane shock configuration which arises in a dusty gas and determine the dependence of the flow field on the parameters of interest. Since the solutions of the problems of interest are described by a many-parameter family of functions, no attempt is made to compile comprehensively the appropriate numbers. Instead, a few examples are worked out and the integrations which lead to detailed results are specified.

2. THE SHOCK PROBLEM

Consider a homogeneous mixture of a gas and small solid particles which is moving with the uniform velocity u_0 as in figure 1. Let the flow configuration be one-dimensional and let the sound speed a_0 of the gas be less than u_0. It can be anticipated that a compressive change of state can occur without spoiling the one-dimensionality of the flow; in particular, when such a compressive change occurs, the first event will be a compression and velocity decrease in the gas consistent with the usual shock relations for the gas; this compression occurs in a very few mean free paths and the solid particles cannot be so quickly decelerated that they could modify these relations. Behind this conventional shock (we shall refer to it as the gas shock) the velocity of the gas is smaller than that of the dust and the dust will then be decelerated. The dust will also accept heat from the gas since the gas temperature has been increased above that of the ambient mixture by the gas shock. Typical distributions of the gas velocity $u(x)$ and the dust velocity $v(x)$ are shown qualitatively in figure 1.

The flow configuration far downstream of the gas shock will be a steady one in which the gas and dust will achieve the same velocities and temperatures. This final state can be computed very simply in the same way that the state following a gas shock is computed. We first write the equations implying the conservation of momentum and energy in the steady one-dimensional flow of this composite 'fluid'. In the statements of these laws we ignore the partial pressure of the dust and we restrict our analysis to shocks which are not so intense that any evaporation of the dust takes place. We also require the shock intensity to be one for which the state law

$$p = \rho RT$$

is a satisfactory approximation. With the notation, $\rho(x)$ = gas density, $m = \rho u$ = const., $\eta(x)$ = dust mass per unit of volume, $n = \eta v$ = const.,

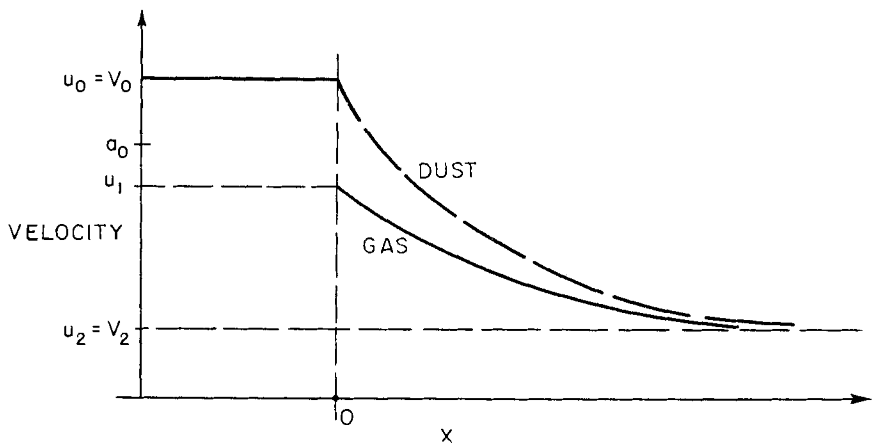

Figure 1. Qualitative gas and dust velocity distributions for the steady plane flow of a dust-gas mixture.

$T(x)$ = gas temperature, $\tau(x)$ = dust temperature (we ignore temperature variations within the particle), and $p(x)$ = gas pressure, these conservation laws are:

$$mu + nv + p = \text{const.,} \tag{1}$$

$$\tfrac{1}{2}mu^2 + \tfrac{1}{2}nv^2 + mc_p T + nc\tau = \text{const.} \tag{2}$$

The quantities c_p and c are, respectively, the specific heat at constant pressure of the gas and the specific heat of the solid material. Since ρu is constant, the equation of state can be written

$$pu = mRT. \tag{3}$$

In the initial state which we characterize by the subscript zero and in the final state (subscript 2), $u = v$ and $T = \tau$. Equations (1), (2) and (3) then imply that

$$(m+n)u_2 + mRT_2/u_2 = (m+n)u_0 + mRT_0/u_0 \tag{4}$$

and

$$\tfrac{1}{2}(m+n)u_2^2 + (mc_p + nc)T_2 = \tfrac{1}{2}(m+n)u_0^2 + (mc_p + nc)T_0. \tag{5}$$

Thus
$$u_2 u_0 = \frac{2m\sigma R}{(m+n)(\sigma+1)} T_0 + \frac{\sigma-1}{\sigma+1} u_0^2, \tag{6}$$
where
$$\sigma = \frac{\gamma + nc/(mc_p)}{1+nc/(mc_p)}$$
and $\gamma = c_p/c_v$. However, u_1, the gas speed just after the gas shock, is given by
$$u_1 u_0 = (a^*)^2 = \frac{2\gamma R}{\gamma+1} T_0 + \frac{\gamma-1}{\gamma+1} u_0^2. \tag{7}$$

A comparison of (6) and (7) shows that the final state and the state just after the gas shock will differ increasingly with increasing values of $nc/(mc_p)$, the ratio of the heat capacity per unit volume of the dust to that of the gas.

The manner in which the state variables vary with x is governed by equations (1), (2) and (3), and by the specification of the mechanism whereby momentum and energy are transferred from one medium to the other. In view of the variety of particle shapes and sizes to be anticipated in problems of interest, it would be optimistic indeed to attempt a description of these transfer processes with the size and shape dependence accounted for in detail. It is more profitable to write down the macroscopic rules:
$$\lambda v v_x = \tfrac{1}{2} C_D l^2 \rho (u-v)^2, \tag{8}$$
$$\lambda c v \tau_x = Nuk(T-\tau) l. \tag{9}$$

These state that the knowledge of the force on a particle of mass λ and 'radius' l requires only the specification of a drag coefficient C_D and that the corresponding knowledge of the heat transfer to such a particle requires the specification of a Nusselt number Nu. In general, C_D and Nu depend on the geometry of the particles, the Reynolds number associated with the motion of the dust relative to that of the gas (i.e. $\rho(u-v)l/\mu$, where μ is the gas viscosity), and the Mach number M of that relative flow ($M = (u-v)/a$, where a is the gas sound speed). However, the Mach number dependence is important only when M is greater than unity over an important portion of the flow field. In view of the restrictions on the flow speed specified above and the increase in sound speed across the gas shock*, the neglect of a Mach number dependence of C_D and Nu does not imply any significant further restriction. Furthermore, the ratio of equations (8) and (9), namely
$$c\frac{d\tau/dx}{dv/dx} = \frac{2Nuk(T-\tau)}{C_D l \rho (u-v)^2}, \tag{10}$$
involves only the ratio Nu/C_D. It is an empirical fact that, for Reynolds numbers Re up to several thousand, $Nu \simeq KC_D Re$, where K is a constant of order unity. When this fact is used, (10) becomes
$$\frac{\tau_x}{v_x} = \beta \frac{T-\tau}{u-v}, \tag{11}$$

* Note that, with $\gamma = 1{\cdot}4$, this Mach number is only of order 2 for very strong shocks.

where $\beta = 2Kk/(c\mu)$. In particular, if the solid material is silica, the gas is air and we use the values of K which are appropriate for spherical or short cylindrical particles, $\beta = 1$ is an excellent estimate.

It is now convenient to regard v as the independent variable of our problem, to note that equations (1), (2) and (3) define T and τ as given functions of u and v, and to regard $x(v)$ as a function to be determined. When this is done, equation (11) can be written as

$$\frac{d\tau(u[v],v)}{dv} = \beta\,\frac{T[u(v),v]-\tau[u(v),v]}{u(v)-v}. \tag{12}$$

Since T and τ are known functions of their arguments, equation (12) is an ordinary differential equation from which we may determine $u(v)$ once a suitable boundary condition is prescribed. The initial condition which must be used requires that $u(v_0) = u_1$, the quantity defined in equation (7). This merely states that just behind the gas shock $u = u_1$ and $v = u_0$. Equation (12) has a singularity (a focus) at the point (u_2, u_2).

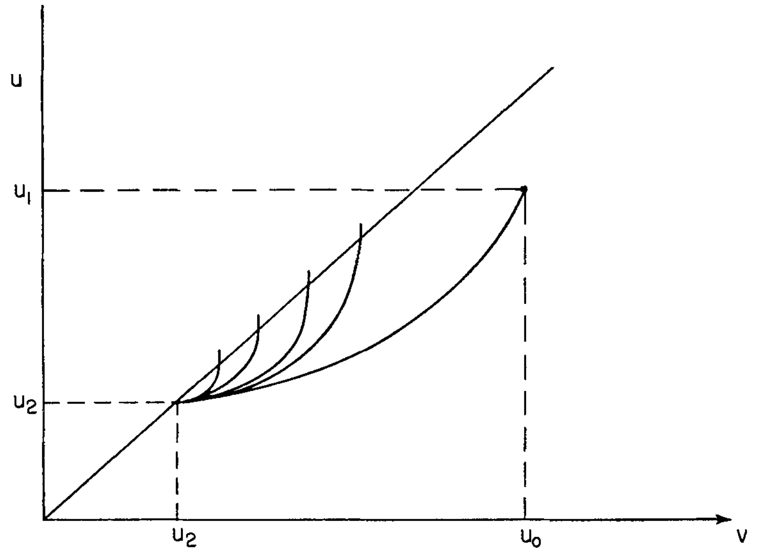

Figure 2. The integral curves of equation (12).

In figure 2 some of the integral curves are sketched qualitatively. That curve which extends from the point (u_1, u_0) to the point (u_2, u_2) is the desired solution of equation (12). That this curve can be obtained accurately by any sensible numerical procedure is readily seen. The nature of the singularity and the fact that T and τ are quadratic in u and v assure a smooth solution curve. Once $u(v)$ has been determined, equation (8) may be integrated directly to give $x(v)$.

It would require an elaborate table of numbers to specify in detail the results of these integrations (e.g. $u(x)$) with regard to their detailed dependence on the independent parameters m/n, c/c_p, $\rho_0 v_0 a/\mu_0$, $u_0/c_p T_0$,

γ, and β. Instead we shall display a few integral curves, $u(v)$, and record the dimensionless forms of the equations which would be convenient in computing further results.

3. The dimensionless equations and results

Since no exhaustive details have been recorded, we compile in this section a dimensionless formulation of the problem which it is convenient to use for computational purposes. Let

$$w = u/a_0, \qquad z = v/a_0, \qquad \theta = T/T_0, \qquad \phi = \tau/T_0, \qquad \delta = c/c_p,$$
$$\epsilon = n/m, \qquad A = (1+\epsilon)w_0 + (\gamma w_0)^{-1}, \qquad B = (1+\epsilon)w_0^2 + (\gamma-1)^{-1}.$$

Quantities with subscript zero characterize the initial state, those with subscript unity the state immediately behind the gas shock, and subscript 2 the final state. β, as defined in § 2, depends on δ, the geometric constant K, and the Prandtl number $c_p \mu/k$. σ defined as in § 2 becomes

$$\sigma = (\gamma+\epsilon\delta)/(1+\epsilon\delta).$$

With the substitution of these dimensionless quantities, equations (1), (2) and (3) become:

$$\theta(w,z) = \gamma w[A - w - \epsilon z], \tag{13}$$

$$\phi(w,z) = \frac{(\gamma-1)}{\epsilon\delta}\left[B - \frac{\theta}{\gamma-1} - \tfrac{1}{2}\epsilon z^2 - \tfrac{1}{2}w^2\right]. \tag{14}$$

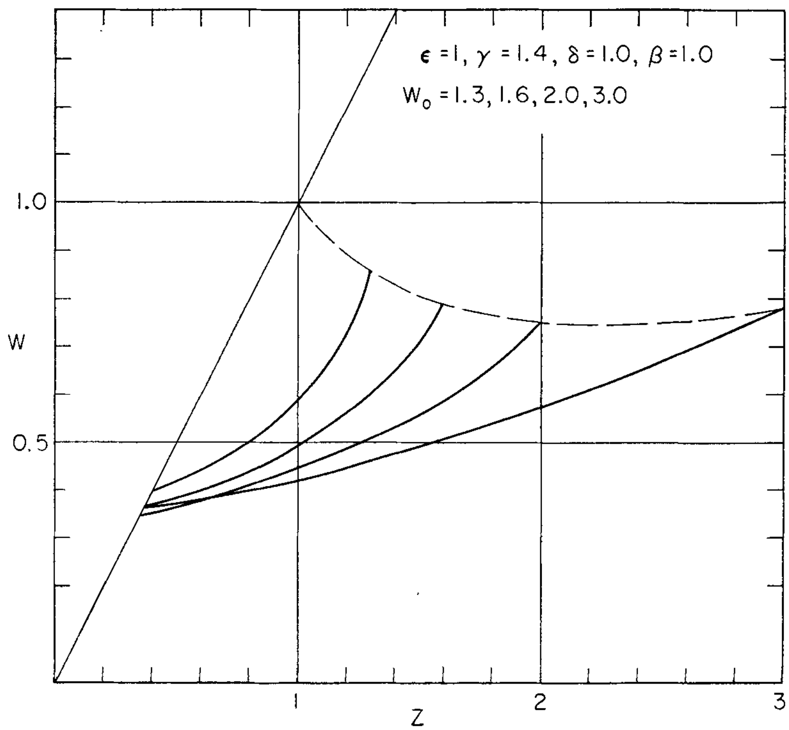

Figure 3. Integral curves of equation (16) with $\epsilon = 1$.

Equation (12) becomes

$$\frac{d\phi}{dz} = \beta \frac{\theta - \phi}{w - z}, \tag{15}$$

and, using equations (13) and (14), this reduces to

$$(w-z)[\gamma A - (\gamma+1)w - \gamma\epsilon z]w'(z) = \epsilon(w-z)[\gamma w - (\gamma-1) - z] - \\ - \beta\epsilon\,\delta[\theta(w,z) - \phi(w,z)]. \tag{16}$$

With the initial condition,

$$w(w_0) = w_1 = \frac{2w_0^{-1} + (\gamma-1)w_0}{\gamma+1},$$

equation (16) can be integrated to give $w(z)$ in the interval $w_0 > z > w_2$. Note that $w(z)$ depends on the independently chosen parameters γ, ϵ, δ, β, and w_0.

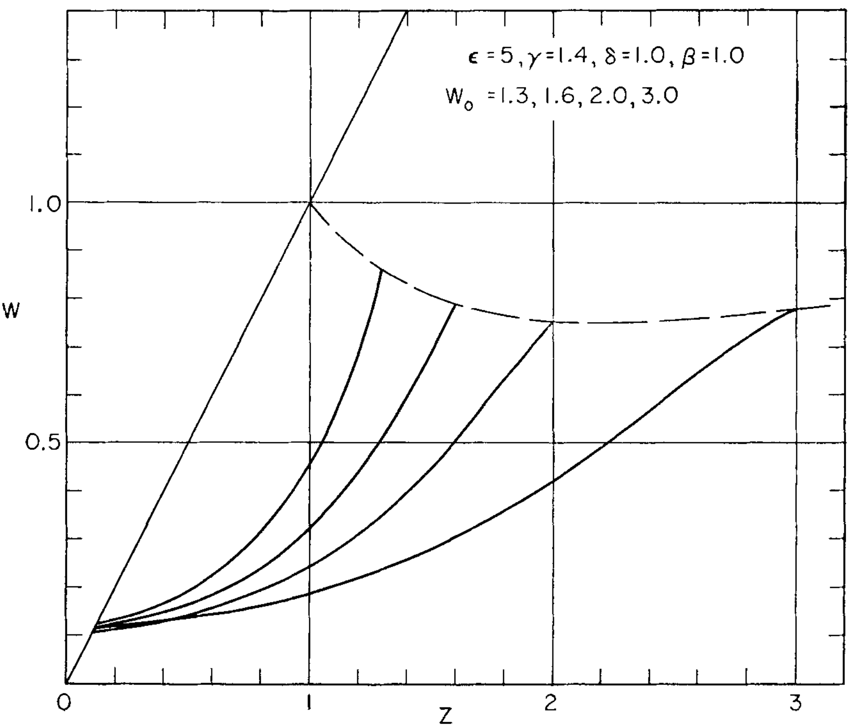

Figure 4. Integral curves of equation (16) with $\epsilon = 5$.

Equation (8) can now be written as

$$x(z) = \frac{2\lambda u_0}{\mu_0 l} \int_{w_0}^{z} \frac{z\,dz}{(w-z) Re\, C_D(Re)}. \tag{17}$$

Here, λ must be interpreted as the average mass per particle and, because of the temperature-dependent gas properties and the variable $(w-z)$, *Re* is a function of z. For a given dust substance, λ is proportional to l^3

and for moderate Reynolds numbers C_D is proportional to $(Re)^{-1/2}$. Thus a measure of the thickness of the transition region is given by

$$X = \lambda(u_0^2/\mu_0 l)(u_0 - u_1)Re^{1/2}, \tag{18}$$

where Re is evaluated at the initial state just behind the gas shock. Since X is just u_0^2 times the mass per particle divided by the drag per particle at the initial state (i.e. u_0 times the time required to decelarate an amount u_0 under the initial drag) the result is not surprising. If the dust were composed of 1-micron particles of silica, for example, and the initial air flow had Mach number 2 and a sea-level stagnation state, X would be about 1 cm. For 10-micron silica, the estimate becomes 30 cm.

A few of the integral curves of equation (16) are drawn in figures 3 and 4 for the values of the parameters specified in the diagrams.